A UNIFIED APPROACH TO LOCAL COHOMOLOGY MODULES USING SERRE CLASSES

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ABSTRACT. This paper discusses the connection between the local cohomology modules and the Serre classes of R-modules. Such connection provided a common language for expressing some results about the local cohomology R-modules, that has appeared in different papers.

1. INTRODUCTION

Throughout this paper R is a commutative Noetherian ring, $\mathfrak{a}$ an ideal of R and M an R-module.

The proofs of some results concerning local cohomology modules indicate that these proofs apply to certain subcategories of R-modules that are closed under taking extensions, submodules and quotients. It should be noted that these kind of subcategories of R-modules are called "Serre classes". In this paper, "$\mathcal{S}$" stands for a "Serre class". The aim of the present paper is to show some results of local cohomology modules are remind true for any Serre classes. As a general reference for local cohomology, we refer the reader to the text book [BS].

Our paper is divided into three sections. In Section 2, we prove the following theorem:

Theorem A. Let $s \in \mathbb{N}_0$ and M be an R-module such that $\operatorname{Ext}^s_R(R/\mathfrak{a}, M) \in \mathcal{S}$. If $\operatorname{Ext}^j_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M)) \in \mathcal{S}$ for all $i < s$ and all $j \geq 0$, then $\operatorname{Hom}_R(R/\mathfrak{a}, H^s_{\mathfrak{a}}(M)) \in \mathcal{S}$.

One can see that the subcategories of, finitely generated R-modules, minimax R-modules, minimax and $\mathfrak{a}$-cofinite R-modules, weakly Laskerian R-modules and Matlis reflexive R-modules are examples of Serre classes. So, we can deduce from Theorem A the main results of [KS], [BL] and [DM, Corollary 2.7], [LSY, Corollary 2.3], [BN, Lemma 2.2], [AKS, Theorem 1.2], see Corollary 2.4, Corollary 2.5, Corollary 2.6, Corollary 2.7, Corollary 2.8 and Corollary 2.10.

In Section 3, we investigate the notation $\operatorname{cd}_{\mathcal{S}}(\mathfrak{a}, M)$ as the supremum of the integers i such that $H^i_{\mathfrak{a}}(M) \notin \mathcal{S}$. We prove that:

Theorem B. Let M and N be finitely generated R-modules. Then the following hold:

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i) Let $t > 0$ be an integer. If N has finite Krull dimension and $H^j_{\mathfrak{a}}(N) \in \mathcal{S}$ for all $j > t$, then $H^t_{\mathfrak{a}}(N)/\mathfrak{a}H^t_{\mathfrak{a}}(N) \in \mathcal{S}$.
ii) If $\operatorname{Supp} N \subseteq \operatorname{Supp} M$, then $\operatorname{cd}_{\mathcal{S}}(\mathfrak{a}, N) \leq \operatorname{cd}_{\mathcal{S}}(\mathfrak{a}, M)$.

If $\mathcal{S}$ is equal to the zero class or the class of Artinian R-modules, then we can obtain the results of [DNT, Theorem 2.2], [DY, Theorem 2.3] and [ADT, Theorem 3.3].

As an application we show that:

Theorem C. Let M be a finitely generated R-module. Then the following hold:

i) If $1 < d := \dim M < \infty$, then $\frac{H^{d-1}_{\mathfrak{a}}(M)}{\mathfrak{a}^n H^{d-1}_{\mathfrak{a}}(M)}$ has finite length for any $n \in \mathbb{N}$.
ii) If $(R, \mathfrak{m})$ is a local ring of Krull dimension less than 3, then $\operatorname{Hom}_R(R/\mathfrak{m}, H^i_{\mathfrak{a}}(M))$ is a finitely generated R-module for all i.

2. Serre classes and common results on local cohomology modules

We need the following observation in the sequel.

Lemma 2.1. *Let $M \in \mathcal{S}$ and N a finitely generated R-module. Then $\operatorname{Ext}^j_R(N, M) \in \mathcal{S}$ and $\operatorname{Tor}^R_j(N, M) \in \mathcal{S}$ for all $j \geq 0$.*

Proof. We only prove the assertion for the Ext modules and the proof for the Tor modules is similar. Let $F_\bullet : \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$ be a finite free resolution of N. If $F_i = R^{n_i}$ for some integer n_i, then $\operatorname{Ext}^i_R(N, M) = H^i(\operatorname{Hom}_R(F_\bullet, M))$ is a subquotient of M^{n_i}. Since $\mathcal{S}$ is a Serre class, it follows that $\operatorname{Ext}^i_R(N, M) \in \mathcal{S}$ for all $i \geq 0$. $\square$

The following is one of the main result of this section.

Theorem 2.2. *Let $s \in \mathbb{N}_0$ and M be an R-module such that $\operatorname{Ext}^s_R(R/\mathfrak{a}, M) \in \mathcal{S}$. If $\operatorname{Ext}^j_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M)) \in \mathcal{S}$ for all $i < s$ and all $j \geq 0$, then $\operatorname{Hom}_R(R/\mathfrak{a}, H^s_{\mathfrak{a}}(M)) \in \mathcal{S}$.*

Proof. We use induction on s. From the isomorphism $\operatorname{Hom}_R(\frac{R}{\mathfrak{a}}, M) \cong \operatorname{Hom}_R(\frac{R}{\mathfrak{a}}, \Gamma_{\mathfrak{a}}(M))$, the case $s = 0$ follows. Now suppose inductively that $s > 0$ and that the assertion holds for $s - 1$. Let $L = M/\Gamma_{\mathfrak{a}}(M)$. Then there exists the exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(M) \longrightarrow M \longrightarrow L \longrightarrow 0.$$

This sequence induces the exact sequences

$$\operatorname{Ext}^j_R(R/\mathfrak{a}, M) \longrightarrow \operatorname{Ext}^j_R(R/\mathfrak{a}, L) \longrightarrow \operatorname{Ext}^{j+1}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M))$$

for all $j \geq 0$. On the other hand, we have $H^i_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{a}}(L)$ for all $i \geq 1$ and $\Gamma_{\mathfrak{a}}(L) = 0$. Also, by our assumption we have $\operatorname{Ext}^j_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)) \in \mathcal{S}$ for all $j \geq 0$. Hence we can

replace M by $M/\Gamma_{\mathfrak{a}}(M)$. Therefore $\Gamma_{\mathfrak{a}}(M)=0$. Let $E_R(M)$ be an injective envelope of M. Then we have the exact sequence

$$0 \longrightarrow M \longrightarrow E_R(M) \longrightarrow N \longrightarrow 0.$$

Since $\Gamma_{\mathfrak{a}}(E_R(M)) = E_R(\Gamma_{\mathfrak{a}}(M)) = 0$, we have $H^i_{\mathfrak{a}}(N) = H^{i+1}_{\mathfrak{a}}(M)$ for all $i \geq 0$. The fact $\operatorname{Hom}_R(R/\mathfrak{a}, E_R(M)) = 0$ implies that $\operatorname{Ext}^j_R(R/\mathfrak{a}, N) \cong \operatorname{Ext}^{j+1}_R(R/\mathfrak{a}, M)$ for all $j \geq 0$. So N satisfies our induction hypothesis. Therefore $\operatorname{Hom}_R(R/\mathfrak{a}, H^{s-1}_{\mathfrak{a}}(N)) \in \mathcal{S}$. The assertion follows from $H^s_{\mathfrak{a}}(M) \cong H^{s-1}_{\mathfrak{a}}(N)$. □

Corollary 2.3. *Let assumptions be as in Theorem 2.2. Let $N \subseteq H^s_{\mathfrak{a}}(M)$ be such that $\operatorname{Ext}^1(R/\mathfrak{a}, N) \in \mathcal{S}$. Then $\operatorname{Hom}_R(R/\mathfrak{a}, H^s_{\mathfrak{a}}(M)/N) \in \mathcal{S}$.*

Proof. The assertion follows from the long Ext exact sequence, induced by

$$0 \to N \to H^s_{\mathfrak{a}}(M) \to H^s_{\mathfrak{a}}(M)/N \to 0. \ \square$$

The categories of, finitely generated R-modules, minimax R-modules [BN, Lemma 2.1], weakly Laskerian R-modules [DM, Lemma 2.3] and Matlis reflexive R-modules, are examples of Serre classes. Hartshorne defined a module M to be $\mathfrak{a}$-cofinite, if $\operatorname{Supp}_R M \subseteq \operatorname{V}(\mathfrak{a})$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ are finitely generated module for all i, see [Har2]. By [M, Corollary 4.4] the class of $\mathfrak{a}$-cofinite minimax modules is a Serre class of the category of R-modules. Consequently, we can deduce the following results from Theorem 2.2 and Corollary 2.3. We denote the set of associated primes of M by $\operatorname{Ass}_R(M)$. Note that $\operatorname{Ass}_R(\operatorname{Hom}_R(R/\mathfrak{a}, M)) = \operatorname{Ass}_R(M)$, for all $\mathfrak{a}$-torsion R-modules M.

Khashyarmanesh and Salarian in [KS] proved the following theorem by the concept of $\mathfrak{a}$-filter regular sequences:

Corollary 2.4. *Let M be a finitely generated R-module and t an integer. Suppose that the local cohomology modules $H^i_{\mathfrak{a}}(M)$ are finitely generated for all $i < t$. Then $\operatorname{Ass}_R(H^t_{\mathfrak{a}}(M))$ is finite.*

On the other hand, Brodmann and Lashgari [BL] generalized this by the basic homological algebraic methods.

Corollary 2.5. *Let M be a finitely generated R-module and t an integer. Suppose that the local cohomology modules $H^i_{\mathfrak{a}}(M)$ are finitely generated for all $i < t$. Then $\operatorname{Ass}_R(H^t_{\mathfrak{a}}(M)/N)$ is finite, for any finitely generated submodule N of $H^i_{\mathfrak{a}}(M)$.*

Recall that, from [DM], an R-module M is weakly Laskerian R-module, if any quotient of M has a finitely many associated primes. Divaani-Aazar and Mafi in [DM, Corollary 2.7] proved by the spectral sequences technics:

Corollary 2.6. *Let M be a weakly Laskerian R-module and t an integer such that $H^i_{\mathfrak{a}}(M)$ is weakly Laskerian modules for all $i < t$. Then $\mathrm{Ass}_R(H^t_{\mathfrak{a}}(M))$ is finite.*

Recall that an R-module M is minimax if there is a finitely generated submodule N of M such that M/N is Artinian, see [Z] and [R].

Corollary 2.7. *(see* [LSY, Corollary 2.3]*). Let M be a minimax R-module. Let t be a non-negative integer such that $H^i_{\mathfrak{a}}(M)$ is a minimax R-module for all $i < t$. Let N be a submodule of $H^t_{\mathfrak{a}}(M)$ such that $\mathrm{Ext}^1_R(R/\mathfrak{a}, N)$ is minimax. Then $\mathrm{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M)/N))$ is minimax. In particular $H^t_{\mathfrak{a}}(M)/N$ has finitely many associated primes.*

The following is a key lemma of [BN, Lemma 2.2]. In fact it is true without $\mathfrak{a}$-cofinite condition, see [BN, Theorem 3.2].

Corollary 2.8. *(see* [BN, Lemma 2.2]*). Let M be a finitely generated R-module. Let t be a non-negative integer such that $H^i_{\mathfrak{a}}(M)$ are minimax and $\mathfrak{a}$-cofinite R-modules for all $i < t$. Then $\mathrm{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M))$ is finitely generated and as a consequence it has finitely many associated primes.*

Proof. Set $\mathcal{S}$ be the class of $\mathfrak{a}$-cofinite and minimax modules. From Theorem 2.2, $\mathrm{Hom}_R(R/\mathfrak{a}, H^s_{\mathfrak{a}}(M))$ is a minimax and $\mathfrak{a}$-cofinite R-module. Therefore we get that $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{Hom}_R(R/\mathfrak{a}, H^s_{\mathfrak{a}}(M))) \cong \mathrm{Hom}_R(R/\mathfrak{a}, H^s_{\mathfrak{a}}(M))$ is finitely generated. $\square$

Corollary 2.9. *Let M a finitely generated R-module and $\mathcal{S}$ a Serre class that contains all finitely generated R-modules. Let t be a non-negative integer such that $H^i_{\mathfrak{a}}(M) \in \mathcal{S}$ for all $i < t$. Then $\mathrm{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M)) \in \mathcal{S}$.*

An immediate consequence of Corollary 2.9 is the following:

Corollary 2.10. *(see* [AKS, Theorem 1.2]*) Let M be a finitely generated R-module. Let t be a non-negative integer such that $H^i_{\mathfrak{a}}(M)$ is finitely generated for all $i < t$. Then $\mathrm{Hom}_R(R/\mathfrak{a}, H^t_{\mathfrak{a}}(M))$ is a finitely generated R-module and as a consequence it has finitely many associated primes.*

In the proof of Theorem 2.12, we will use the following lemma.

Lemma 2.11. *Let $(R, \mathfrak{m})$ be a local ring and $\mathcal{S}$ a non-zero Serre class. Let $\mathcal{FL}$ be the class of finite length R-modules. Then $\mathcal{FL} \subseteq \mathcal{S}$.*

Proof. Since $\mathcal{S}$ is non-zero, there exists a non-zero R-module $L \in \mathcal{S}$. Let $0 \neq m \in L$. Then $Rm \in \mathcal{S}$. From the natural epimorphism $Rm \cong R/(0 :_R m) \twoheadrightarrow R/\mathfrak{m}$ we obtained that $R/\mathfrak{m} \in \mathcal{S}$. Let $M \in \mathcal{FL}$ and set $\ell := \ell_R(M)$. By induction on ℓ, we show that $M \in \mathcal{S}$. For the cases $\ell = 0, 1$, we have nothing to prove. Now suppose inductively, $\ell > 0$ and

the result has been proved for each finite length R-module N, with $\ell_R(N) \leq \ell - 1$. By definition there is following chain of R-submodules of M:

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\ell = M$$

such that $M_j/M_{j-1} \cong R/\mathfrak{m}$. Now the exact sequence

$$0 \longrightarrow M_{\ell-1} \longrightarrow M \longrightarrow R/\mathfrak{m} \longrightarrow 0,$$

completes the proof. $\square$

Now we are ready to prove the second main result of this section.

Theorem 2.12. *Let $(R, \mathfrak{m})$ be a local ring, $\mathcal{S}$ a non-zero Serre class and M a finitely generated R-module. Let t be a non-negative integer such that $H^i_{\mathfrak{a}}(M) \in \mathcal{S}$ for all $i < t$. Then $\operatorname{Hom}_R(R/\mathfrak{m}, H^t_{\mathfrak{a}}(M)) \in \mathcal{S}$.*

Proof. We do induction on t. If $t = 0$, then $\operatorname{Hom}_R(R/\mathfrak{m}, H^0_{\mathfrak{a}}(M))$ has finite length. So by Lemma 2.11, $\operatorname{Hom}_R(R/\mathfrak{m}, H^0_{\mathfrak{a}}(M)) \in \mathcal{S}$. Now suppose inductively, $t > 0$ and the result has been proved for all integer smaller than t. We have $H^i_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M))$ for all $i > 0$. Hence we may assume that M is $\mathfrak{a}$-torsion free. Take $x \in \mathfrak{a} \setminus \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R M} \mathfrak{p}$. From the exact sequence

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$$

we deduced the long exact sequence of local cohomology modules, which shows that $H^j_{\mathfrak{a}}(M/xM) \in \mathcal{S}$ for all $j < t-1$. Thus, $\operatorname{Hom}_R(R/\mathfrak{m}, H^{t-1}_{\mathfrak{a}}(M/xM)) \in \mathcal{S}$.

Now, consider the long exact sequence

$$\cdots \longrightarrow H^{t-1}_{\mathfrak{a}}(M/xM) \longrightarrow H^t_{\mathfrak{a}}(M) \xrightarrow{x} H^t_{\mathfrak{a}}(M) \longrightarrow \cdots,$$

which induces the following exact sequence

$$0 \longrightarrow H^{t-1}_{\mathfrak{a}}(M)/xH^{t-1}_{\mathfrak{a}}(M) \longrightarrow H^{t-1}_{\mathfrak{a}}(M/xM) \longrightarrow (0 :_{H^t_{\mathfrak{a}}(M)} x) \longrightarrow 0.$$

From this we get the following exact sequence

$$\operatorname{Hom}_R(\frac{R}{\mathfrak{m}}, H^{t-1}_{\mathfrak{a}}(\frac{M}{xM})) \longrightarrow \operatorname{Hom}_R(\frac{R}{\mathfrak{m}}, (0 :_{H^t_{\mathfrak{a}}(M)} x)) \longrightarrow \operatorname{Ext}^1_R(\frac{R}{\mathfrak{m}}, \frac{H^{t-1}_{\mathfrak{a}}(M)}{xH^{t-1}_{\mathfrak{a}}(M)}).$$

By Lemma 2.1, $\operatorname{Ext}^1_R(\frac{R}{\mathfrak{m}}, \frac{H^{t-1}_{\mathfrak{a}}(M)}{xH^{t-1}_{\mathfrak{a}}(M)}) \in \mathcal{S}$. Therefore, $\operatorname{Hom}_R(R/\mathfrak{m}, (0 :_{H^t_{\mathfrak{a}}(M)} x)) \in \mathcal{S}$. The following completes the proof:

$$\begin{aligned} \operatorname{Hom}_R(\tfrac{R}{\mathfrak{m}}, H^t_{\mathfrak{a}}(M)) &\cong \operatorname{Hom}_R(R/\mathfrak{m} \otimes_R R/xR, H^t_{\mathfrak{a}}(M)) \\ &\cong \operatorname{Hom}_R(R/\mathfrak{m}, (0 :_{H^t_{\mathfrak{a}}(M)} x)). \ \square \end{aligned}$$

Example 2.13. In Theorem 2.12, the assumption $\mathcal{S} \neq \{0\}$ is necessary. To see this, let $(R, \mathfrak{m})$ be a local Gorenstein ring of positive dimension d. Then $H^i_{\mathfrak{m}}(R) = 0$ for $i < d$. But $\mathrm{Hom}_R(R/\mathfrak{m}, H^d_{\mathfrak{m}}(R)) \cong \mathrm{Hom}_R(R/\mathfrak{m}, E) \cong R/\mathfrak{m} \neq 0$, where E is an injective envelope of $R/\mathfrak{m}$.

As an immediate result of Theorem 2.12 (or Corollary 2.10) we have the following corollary.

Corollary 2.14. *Let $(R, \mathfrak{m})$ be a local ring and M a finitely generated R-module. Let t be an non-negative integer such that $H^i_{\mathfrak{a}}(M)$ is a finitely generated R-module for all $i < t$. Then* $\mathrm{Hom}_R(R/\mathfrak{m}, H^t_{\mathfrak{a}}(M))$ *is a finitely generated R-module.*

Let $(R, \mathfrak{m})$ be a local ring. The third of Huneke's four problems in local cohomology [Hu] is to determine when $H^i_{\mathfrak{a}}(M)$ is Artinian for a finitely generated R-module M. The mentioned problem may be separated into two subproblems:

i) When is $\mathrm{Supp}_R(H^i_{\mathfrak{a}}(M)) \subseteq \{\mathfrak{m}\}$?
ii) When is $\mathrm{Hom}_R(R/\mathfrak{m}, H^i_{\mathfrak{a}}(M))$ finitely generated?

Huneke formalized the following conjecture, see [Hu, Conjecture 4.3].

Conjecture. Let $(R, \mathfrak{m})$ be a regular local ring and $\mathfrak{a}$ be an ideal of R. For all i, $\mathrm{Hom}_R(R/\mathfrak{m}, H^i_{\mathfrak{a}}(R))$ is finitely generated.

It is known that if R is an unramified regular local ring, then $\mathrm{Hom}_R(R/\mathfrak{m}, H^i_{\mathfrak{a}}(R))$ is finitely generated, for all i (see [HS], [L1], [L2]). The first example of a local cohomology module with an infinite dimensional socle was given in [Har2] by Hartshorne. The Hartshorne's famous example is a three dimensional local ring.

As the first application, the following provides a positive answer of the conjecture, for all local rings of Krull dimension less than 3.

Corollary 2.15. *Let $(R, \mathfrak{m})$ be a local ring of dimension less than 3, and M a finitely generated R-module. Then* $\mathrm{Hom}_R(R/\mathfrak{m}, H^i_{\mathfrak{a}}(M))$ *is a finitely generated R-module for all i.*

Proof. First assume that $\dim R = 2$. The cases $i = 0$ and $i > 2$ are trivial, since $H^0_{\mathfrak{a}}(M)$ is finitely generated R-module and $H^i_{\mathfrak{a}}(M) = 0$ for all $i > 2$. Note that $H^2_{\mathfrak{a}}(M)$ is an Artinian R-module. Therefore, $\mathrm{Hom}_R(R/\mathfrak{m}, H^2_{\mathfrak{a}}(M))$ is a finitely generated R-module. In the case $i = 1$ one can get the desired result from Corollary 2.14.

If $\dim R \leq 1$ we can obtain the desired result in similar way. $\square$

Remark 2.16. Let n be an integer grater than 2. Then [MV, Theorem 1.1] and the discussion before than [MV, Question 2.1], provided an n-dimensional regular local ring $(R, \mathfrak{m})$ and a finitely generated R-module M such that $\mathrm{Hom}_R(R/\mathfrak{m}, H^t_{\mathfrak{a}}(M))$ is not finitely generated R-module, for some $t \in \mathbb{N}$ and some ideal $\mathfrak{a} \lhd R$.

3. Serre cohomological dimension

In the proof of the following theorem we use the method of the proof of [ADT, Theorem 3.3].

Theorem 3.1. *Let $\mathfrak{a}$ be an ideal of R and M a weakly Laskerian R-module of finite Kryll dimension. Let $t > 0$ be an integer. If $H^j_{\mathfrak{a}}(M) \in \mathcal{S}$ for all $j > t$, then $H^t_{\mathfrak{a}}(M)/\mathfrak{a}H^t_{\mathfrak{a}}(M) \in \mathcal{S}$.*

Proof. We use induction on $d := \dim M$. The case $d = 0$, is easy, because $H^t_{\mathfrak{a}}(M) = 0$. Now suppose inductively, $\dim M = d > 0$ and the result has been proved for all R-modules of dimension smaller than d. We have $H^i_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{a}}(M/\Gamma_{\mathfrak{a}}(M))$ for all $i > 0$. Also $M/\Gamma_{\mathfrak{a}}(M)$ has dimension not exceeding d. So we may assume that M is $\mathfrak{a}$-torsion free. Let $x \in \mathfrak{a} \setminus \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R M} \mathfrak{p}$. Then M/xM is weakly Laskerian and $\dim M/xM \leq d - 1$. The exact sequence

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0$$

induces the long exact sequence of local cohomology modules, which shows that $H^j_{\mathfrak{a}}(M/xM) \in \mathcal{S}$ for all $j > t$. By induction hypothesis $H^t_{\mathfrak{a}}(M/xM)/\mathfrak{a}H^t_{\mathfrak{a}}(M/xM) \in \mathcal{S}$.

Now, consider the exact sequence

$$H^t_{\mathfrak{a}}(M) \xrightarrow{x} H^t_{\mathfrak{a}}(M) \xrightarrow{f} H^t_{\mathfrak{a}}(M/xM) \xrightarrow{g} H^{t+1}_{\mathfrak{a}}(M),$$

which induces the following two exact sequences

$$H^t_{\mathfrak{a}}(M) \xrightarrow{x} H^t_{\mathfrak{a}}(M) \longrightarrow \operatorname{Im} f \longrightarrow 0,$$

$$0 \longrightarrow \operatorname{Im} f \longrightarrow H^t_{\mathfrak{a}}(M/xM) \longrightarrow \operatorname{Im} g \longrightarrow 0.$$

Therefore we can obtain the following two exact sequences:

$$H^t_{\mathfrak{a}}(M)/\mathfrak{a}H^t_{\mathfrak{a}}(M) \xrightarrow{x} H^t_{\mathfrak{a}}(M)/\mathfrak{a}H^t_{\mathfrak{a}}(M) \longrightarrow \operatorname{Im} f/\mathfrak{a} \operatorname{Im} f \longrightarrow 0,$$

$$Tor^R_1(R/\mathfrak{a}, \operatorname{Im} g) \longrightarrow \operatorname{Im} f/\mathfrak{a} \operatorname{Im} f \longrightarrow H^t_{\mathfrak{a}}(M/xM)/\mathfrak{a}H^t_{\mathfrak{a}}(M/xM) \longrightarrow \operatorname{Im} g/\mathfrak{a} \operatorname{Im} g \longrightarrow 0.$$

Since $x \in \mathfrak{a}$, from a preceding exact sequence, we get that $\operatorname{Im} f/\mathfrak{a} \operatorname{Im} f \cong H^t_{\mathfrak{a}}(M)/\mathfrak{a}H^t_{\mathfrak{a}}(M)$. By Lemma 2.1, we have $Tor^R_1(R/\mathfrak{a}, \operatorname{Im} g) \in \mathcal{S}$. Also $H^t_{\mathfrak{a}}(M/xM)/\mathfrak{a}H^t_{\mathfrak{a}}(M/xM) \in \mathcal{S}$. So $\operatorname{Im} f/\mathfrak{a} \operatorname{Im} f \in \mathcal{S}$. Now the claim follows. $\square$

The second of our applications is the following corollary.

Corollary 3.2. *Let M be a finitely generated R-module of finite Krull dimension $d > 1$. Then $\frac{H^{d-1}_{\mathfrak{a}}(M)}{\mathfrak{a}^n H^{d-1}_{\mathfrak{a}}(M)}$ has finite length for any $n \in \mathbb{N}$.*

Proof. We have $H^{d-1}_{\mathfrak{a}}(M) = H^{d-1}_{\mathfrak{a}^n}(M)$. So it is enough to prove the desired result for $n = 1$. By [M, Proposition 5.1], $H^d_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite and Artinian. Set $\mathcal{S} := \{N : N \textit{ is a } \mathfrak{a}\textit{-cofinite and minimax } R\textit{-module}\}$. In view of Theorem 3.1, we get that the R-module $\frac{H^{d-1}_{\mathfrak{a}}(M)}{\mathfrak{a}H^{d-1}_{\mathfrak{a}}(M)}$ is $\mathfrak{a}$-cofinite. So $\operatorname{Hom}_R(R/\mathfrak{a}, \frac{H^{d-1}_{\mathfrak{a}}(M)}{\mathfrak{a}H^{d-1}_{\mathfrak{a}}(M)}) \cong \frac{H^{d-1}_{\mathfrak{a}}(M)}{\mathfrak{a}H^{d-1}_{\mathfrak{a}}(M)}$ is a finitely generated R-module. Set $\mathcal{S} := \{N : N \textit{ is an Artinian } R\textit{-module}\}$. Again by Theorem 3.1, we get that the R-module $\frac{H^{d-1}_{\mathfrak{a}}(M)}{\mathfrak{a}H^{d-1}_{\mathfrak{a}}(M)}$ is an Artinian R-module. Consequently, the R-module $\frac{H^{d-1}_{\mathfrak{a}}(M)}{\mathfrak{a}H^{d-1}_{\mathfrak{a}}(M)}$ has finite length. □

Example 3.3. In Corollary 3.2, if $t < \dim M - 1$, then it can be seen that $H^t_{\mathfrak{a}}(N)/\mathfrak{a}H^t_{\mathfrak{a}}(N)$ is not necessarily has finite length. To see this, let $R := k[[X_1, \cdots, X_4]]$, $\mathfrak{J}_1 := (X_1, X_2)$, $\mathfrak{J}_2 := (X_3, X_4)$ and $\mathfrak{a} := \mathfrak{J}_1 \cap \mathfrak{J}_2$, where k is a field. By Mayer-Vietoris exact sequence we get that $H^2_{\mathfrak{a}}(R) \cong H^2_{\mathfrak{J}_1}(R) \oplus H^2_{\mathfrak{J}_2}(R)$. Now consider the following isomorphisms

$$\begin{aligned} H^2_{\mathfrak{a}}(R)/\mathfrak{a}H^2_{\mathfrak{a}}(R) &\cong (H^2_{\mathfrak{J}_1}(R)/\mathfrak{a}H^2_{\mathfrak{J}_1}(R)) \oplus (H^2_{\mathfrak{J}_2}(R)/\mathfrak{a}H^2_{\mathfrak{J}_2}(R)) \\ &\cong H^2_{\mathfrak{J}_1}(R/\mathfrak{a}) \oplus H^2_{\mathfrak{J}_2}(R/\mathfrak{a}). \end{aligned}$$

By Hartshorne-Lichtenbaum vanishing Theorem, $H^2_{\mathfrak{J}_1}(R/\mathfrak{a}) \neq 0$. Therefore the cohomological dimension of $R/\mathfrak{a}$ with respect to $\mathfrak{J}_1$ is two. By [Hel, Remark 2.5] the local cohomology $H^2_{\mathfrak{J}_1}(R/\mathfrak{a})$ is not finitely generated. Consequently, $H^2_{\mathfrak{a}}(R)/\mathfrak{a}H^2_{\mathfrak{a}}(R)$ is not finitely generated.

Definition 3.4. Let M be an R-module and $\mathfrak{a}$ an ideal of R. For a Serre class $\mathcal{S}$, we define $\mathcal{S}$-cohomological dimension of M with respect to $\mathfrak{a}$, by $\operatorname{cd}_{\mathcal{S}}(\mathfrak{a}, M) := \sup\{i \in \mathbb{N}_0 : H^i_{\mathfrak{a}}(M) \notin \mathcal{S}\}$.

Theorem 3.5. *Let M and N be finitely generated R-modules such that* $\operatorname{Supp}_R N \subseteq \operatorname{Supp}_R M$. *Then* $\operatorname{cd}_{\mathcal{S}}(\mathfrak{a}, N) \leq \operatorname{cd}_{\mathcal{S}}(\mathfrak{a}, M)$.

Proof. It is enough to show that if $i > \operatorname{cd}_{\mathcal{S}}(\mathfrak{a}, M)$, then $H^i_{\mathfrak{a}}(N) \in \mathcal{S}$. We prove this by descending induction on i with $\operatorname{cd}_{\mathcal{S}}(\mathfrak{a}, M) < i \leq dim(M) + 1$. Note that any non empty Serre class containing the zero module. By Grothendieck's vanishing theorem, in the case $i = \dim M + 1$ we have nothing to prove. Now suppose $\operatorname{cd}_{\mathcal{S}}(\mathfrak{a}, M) < i \leq \dim M$ and we have proved that $H^{i+1}_{\mathfrak{a}}(K) \in \mathcal{S}$ for each finitely generated R-module K with $\operatorname{Supp}_R K \subseteq \operatorname{Supp}_R M$. By theorem of Gruson [V, Theorem 4.1], there is a chain

$$0 = N_0 \subset N_1 \subset \cdots \subset N_\ell = N$$

such that each of the factors N_j/N_{j-1} is a homomorphic image of a direct sum of finitely many copies of M. By using short exact sequences, the situation can be reduced to the case $\ell = 1$. Therefore, for some positive integer n and some finitely generated R-module L, there exists an exact sequence $0 \longrightarrow L \longrightarrow M^n \longrightarrow N \longrightarrow 0$. Thus we have the

following long exact sequence

$$\cdots \longrightarrow H^i_{\mathfrak{a}}(L) \longrightarrow H^i_{a}(M^n) \longrightarrow H^i_{\mathfrak{a}}(N) \longrightarrow H^{i+1}_{\mathfrak{a}}(L) \longrightarrow \cdots .$$

By the inductive assumption $H^{i+1}_{\mathfrak{a}}(L) \in \mathcal{S}$. Since $H^i_{\mathfrak{a}}(M^n) \in \mathcal{S}$ we get that $H^i_{\mathfrak{a}}(N) \in \mathcal{S}$. This completes the inductive step. $\square$

Let $\mathcal{A}$ be the class of Artinin R-modules. Recall that in the literatures the notion $\operatorname{cd}_{\{0\}}(\mathfrak{a}, M)$ is denote by $\operatorname{cd}(\mathfrak{a}, M)$ and $\operatorname{cd}_{\mathcal{A}}(\mathfrak{a}, M)$ by $q_{\mathfrak{a}}(M)$. Here, we record several immediate consequences of Theorem 3.5.

Corollary 3.6. *(see* [DNT, Theorem 2.2]*) Let M and N be finitely generated R-modules such that* $\operatorname{Supp}_R N \subseteq \operatorname{Supp}_R M$. *Then* $\operatorname{cd}(\mathfrak{a}, N) \leq \operatorname{cd}(\mathfrak{a}, M)$.

Corollary 3.7. *Let M be a finitely generated R-module. Then* $\operatorname{cd}_{\mathcal{S}}(\mathfrak{a}, M) = \max\{\operatorname{cd}_{\mathcal{S}}(\mathfrak{a}, R/\mathfrak{p}) : \mathfrak{p} \in \operatorname{Ass}_R M\}$.

Proof. Let $N := \oplus_{\mathfrak{p} \in \operatorname{Ass}_R M} R/\mathfrak{p}$. Then N is finitely generated and $\operatorname{Supp}_R N = \operatorname{Supp}_R M$. In view of Theorem 3.5, $\operatorname{cd}_{\mathcal{S}}(\mathfrak{a}, M) = \operatorname{cd}_{\mathcal{S}}(\mathfrak{a}, N) = \max\{\operatorname{cd}_{\mathcal{S}}(\mathfrak{a}, R/\mathfrak{p}) : \mathfrak{p} \in \operatorname{Ass}_R M\}$. $\square$

Corollary 3.8. *(see* [DY, Theorem 2.3]*) Let M and N be finitely generated R-modules such that* $\operatorname{Supp}_R N \subseteq \operatorname{Supp}_R M$. *Then* $q_{\mathfrak{a}}(N) \leq q_{\mathfrak{a}}(M)$.

We denote by $q(\mathfrak{a})$ the supremum of all integers j for which there is a finitely generated R-module M, with $H^j_{\mathfrak{a}}(M)$ not Artinian. It was proved by Hartshorn [Har1] that $q(\mathfrak{a})$ is the supremum of all integers j for which $H^j_{\mathfrak{a}}(R)$ is not Artinian. The following is a generalization of this result.

Corollary 3.9. $\operatorname{cd}_{\mathcal{S}}(\mathfrak{a}, R) = \sup\{\operatorname{cd}_{\mathcal{S}}(\mathfrak{a}, N) | N$ *is a finitely generated R-module*$\}$. *In particular if $H^j_{\mathfrak{a}}(R) \in \mathcal{S}$ for all $j > \ell$, then $H^j_{\mathfrak{a}}(M) \in \mathcal{S}$ for all $j > \ell$ and all finitely generated R-module M.*

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